

A brief review on geometry and spectrum of graphs

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April 12, 2012

1 Introduction

This is a survey paper. We study the Ricci curvature and spectrum of graphs, as well as the exterior forms and deRahm cohomology on graphs.

2 Geometry and Spectral theory for Graphs

2.1 Ricci curvature on graphs

Let $G = (V, E)$ be a graph, where V is a vertices set and E is the set of edges. For $x, y \in V$, $x \sim y$ means that x is adjacent to y . Let d_x denote the degree of the vertex x . If $d_x < +\infty$ for all $x \in V$, we say that G is a locally finite graph. If d_x is same for every x , we say that the graph is a regular graph. For two vertices x and y , the distance between x and y is the number of edges in the shortest path joining x and y . The diameter of a graph G is the maximum distance between any two vertices of G . We always assume that G is connected, which means that any two vertices of G can be connected by a path in G .

The first definition of Ricci curvature was introduced by Fan Chung Graham and S. T. Yau in 1996[6]. In the course of obtaining a good log-Sobolev inequality, they found the following definition of Ricci curvature to be useful:

We say that a regular graph G has a local k -frame at a vertex x if there exists injective mappings η_1, \dots, η_k from a neighborhood of x into V so that

1. x is adjacent to $\eta_i x$ for $1 \leq i \leq k$;
2. $\eta_i x \neq \eta_j x$ if $i \neq j$.

The graph G is said to be Ricci-flat at x if there is a local k -frame in a neighborhood of x so that for all i ,

$$\bigcup_j (\eta_i \eta_j) x = \bigcup_j (\eta_j \eta_i) x .$$

For a more general definition of Ricci curvature, we need the following.

We first define the Laplace operator on graph without loops and multiple edges. The description in the following can be used for weighted graphs. But for simplicity, we set all weights here equal to 1.

Let $V^R = \{f|f : V \rightarrow R\}$. The Laplace operator Δ of a graph G is

$$\Delta f(x) = \frac{1}{d_x} \sum_{y \sim x} [f(y) - f(x)]$$

for all $f \in V^R$. For graphs, we have

$$|\nabla f(x)|^2 = \frac{1}{d_x} \sum_{y \sim x} [f(y) - f(x)]^2 .$$

We first introduce a bilinear operator $\Gamma : V^R \times V^R \rightarrow V^R$ by

$$\Gamma(f, g)(x) = \frac{1}{2} \{ \Delta(f(x) \cdot g(x)) - f(x) \Delta g(x) - g(x) \Delta f(x) \} .$$

The Ricci curvature operator Γ_2 is defined by iterating the Γ :

$$\Gamma_2(f, g)(x) = \frac{1}{2} \{ \Delta \Gamma(f, g)(x) - \Gamma(f, \Delta g)(x) - \Gamma(g, \Delta f)(x) \} .$$

Definition. The Laplace operator Δ on graphs satisfies the curvature-dimension type inequality $CD(m, K)$ ($m \in (1, +\infty]$) if

$$\Gamma_2(f, f)(x) \geq \frac{1}{m} (\Delta(f(x)))^2 + k(x) \cdot \Gamma(f, f)(x)$$

We call m the dimension of the operator Δ and $k(x)$ the lower bound of the Ricci curvature of the operator Δ .

It is easy to see that for $m < m'$, the operator Δ satisfies $CD(m', k)$ if it satisfies $CD(m, k)$.

Remark. We find that:

$$\begin{aligned} \Delta(|\nabla f(x)|^2) &= \frac{1}{d_x} \sum_{y \sim x} \frac{1}{d_x} \sum_{z \sim y} [f(x) - 2f(y) + f(z)]^2 \\ &\quad - \frac{2}{d_x} \sum_{y \sim x} \frac{1}{d_y} \sum_{z \sim y} [f(x) - 2f(y) + f(z)][f(x) - f(y)] \end{aligned}$$

and

$$\Gamma(f, \Delta f)(x) = \frac{1}{2} \frac{1}{d_y} \sum_{y \sim x} [f(y) - f(x)][\Delta f(y) - \Delta f(x)] .$$

From the definition of the Ricci curvature operator, we obtain:

$$\begin{aligned}\Gamma_2(f, f)(x) &= \frac{1}{4} \frac{1}{d_x} \sum_{y \sim x} \frac{1}{d_y} \sum_{z \sim y} [f(x) - 2f(y) + f(z)]^2 \\ &\quad - \frac{1}{2} \frac{1}{d_x} \sum_{y \sim x} [f(y) - f(x)]^2 \\ &\quad + \frac{1}{2} \left[\frac{1}{d_x} \sum_{y \sim x} (f(y) - f(x)) \right]^2.\end{aligned}$$

We see that the lower bound of the Ricci curvature $k(x)$ only depends on those vertices with distant at most 2 from vertex x . So we can choose the function $f(x)$ to be supported only on those vertices.

- Suppose Δ is the Laplace-Beltrami operator on a m -dimension complete connected Riemannian manifold M .

Bochner's formula indicates that

$$\Gamma_2(f, f)(x) = Ric(\nabla f(x), \nabla f(x)) + \|Hess f(x)\|_2^2,$$

where Ric is the Ricci tensor on M and $\|Hess f\|_2$ is the Hilbert-Schmidt norm of the tensor of the second derivative of f . Since $\|Hess f\|_2^2 \geq \frac{1}{m}(\Delta f)^2$, so we say that Δ satisfies the $CD(m, k)$ if and only if the Ricci curvature at x on M is bounded below by k .

- In 1985, Bakry and Emery[1] considered these notions on metric measure space where the operator Δ is the so-called diffusion operator, that is Δ satisfies the following chain-rule formula: for every C^∞ function Φ on R and every function f ,

$$\Delta \phi(f) = \phi'(f) \Delta f + \phi''(f) \Gamma(f, f).$$

The Laplace operator Δ on graphs does not satisfy the chain-rule formula. But by the following Lemma, it is the correct operator for defining the Ricci curvature operator on graphs.

In the classical case, $\Gamma(f, f) = \frac{1}{2} |\nabla(f)|^2$. On graphs, we also have

Lemma. Suppose G is a locally finite graph and Δ is the Laplace operator on G , then

$$\Gamma(f, f) = \frac{1}{2} |\nabla f|^2.$$

Proof.

$$\begin{aligned}
\Delta(f^2(x)) &= \frac{1}{d_x} \sum_{y \sim x} [f^2(y) - f^2(x)] \\
&= \frac{1}{d_x} \sum_{y \sim x} [f(y) - f(x)] \cdot [f(y) + f(x)] \\
&= \frac{2}{d_x} \sum_{y \sim x} [f(y) - f(x)] \cdot f(x) + \frac{1}{d_x} \sum_{y \sim x} [f(y) - f(x)]^2 \\
&= 2f(x)\Delta f(x) + |\nabla f(x)|^2.
\end{aligned}$$

So

$$\Gamma(f, f) = \frac{1}{2}[\Delta(f)^2 - 2f\Delta f] = \frac{1}{2}|\nabla(f)|^2.$$

□

For the Ricci curvature operator on graphs, we have the following theorem.

Theorem (Y. Lin, S. T. Yau[13]). Suppose G is a locally finite graph and $d = \sup_{x \in V} d_x$ (where d can be infinite). Then we have

$$\Gamma_2(f, f) \geq \frac{1}{2}(\Delta f)^2 + \left(\frac{1}{d} - 1\right)\Gamma(f, f)$$

i.e. the Laplace operator Δ on G satisfies $CD(2, \frac{1}{d} - 1)$.

Remark. The Ricci-flat graph in the sense of Chung and Yau satisfies $CD(\infty, 0)$.

We also can obtain an eigenvalue estimate for the finitely connected graph.

Suppose a function $f(x) \in V^R$ satisfies

$$(-\Delta)f(x) = \lambda f(x),$$

then $f(x)$ is called the eigenfunction of the Laplace operator Δ on G with eigenvalue λ .

We prove the following result which is similar to the classical result of Li and Yau on compact manifold with Ricci curvature bounded below.

Theorem (Y. Lin, S. T. Yau[13]). Suppose G is a connected graph with diameter D , then the non-zero eigenvalue of Laplace operator Δ on G

$$\lambda \geq \frac{1}{dD(\exp(dD + 1) - 1)}.$$

Notice that there is a well-known estimate for the eigenvalue of Laplacian on graphs

$$\lambda \geq \frac{1}{D \text{Vol}G} ,$$

where $\text{Vol}G = \sum_{x \in V} d_x$.

The proof of the above Theorem is similar to the proof of Li and Yau on manifold by using the following gradient estimate

$$\frac{|\nabla f(x)|^2}{(\beta - f(x))^2} \leq d \left(\frac{1}{\beta - 1} \lambda + 1 \right)^2 ,$$

where $\beta > 1$.

For the graph with Ricci curvature bounded below by a positive number, There are the following eigenvalue estimate which is similar to the Lichnerowicz theorem in Riemannian manifold. We can also show the estimate is shape for $m = \infty$ and $m = 2$.

Theorem (Fan Chung, Y. Lin, Y. Liu[4]). Suppose a finite graph G satisfies the $CD(m, k)$ with $k > 0$, then the non-zero eigenvalue of Δ on G

$$\lambda \geq \frac{m}{m-1} k.$$

3 Harnack inequality and eigenvalue estimate on graphs

3.1 Harnack inequality and eigenvalue estimate on graphs

In this section, we will establish the Harnack inequality, as a consequence, we get eigenvalue estimate for graphs with Ricci curvature bounded below by some constants.

The idea is to use the maximum principle similar to the Eucliden and manifold case.

We can prove the following Harnack type inequality. The idea of proof comes from [3] and [2].

Theorem (Fan Chung, Y. Lin, S. T. Yau[5]). Suppose a finite graph G satisfies the $CD(m, k)$, $f \in V^R$ is an eigenfunction of Laplacian Δ with eigenvalue λ . Then the following inequality holds for all $x \in V$.

$$|\nabla f(x)|^2 \leq [(8 - \frac{2}{m})\lambda - 4k] \sup_{z \in V} f^2(x).$$

As a consequence, we can get an eigenvalue estimate for graphs.

Theorem (Fan Chung, Y. Lin, S. T. Yau[5]). Suppose a finite connected graph G satisfies $CD(m, k)$ and λ is a non-zero eigenvalue of Laplace operator Δ on G . Then

$$\lambda \geq \frac{1 + 4kdD^2}{d \cdot (8 - \frac{2}{m}) \cdot D^2}$$

where d is the maximum degree and D denotes the diameter of G .

Remark. This means we still have an eigenvalue lower bound for graphs with Ricci curvature bounded below by some negative number, that is when

$$k > -\frac{1}{4dD^2}.$$

For graph with non-negative Ricci curvature. We have:

Corollary. Suppose a finitely connected graph G satisfies $CD(m, 0)$, then the non-zero eigenvalue of Δ on G

$$\lambda \geq \frac{1}{(8 - \frac{1}{m})dD^2}.$$

Remark. Fan Chung and Yau proved a similar result for so-called Ricci-flat graphs[6]. The Ricci-flat graphs satisfy $CD(\infty, 0)$. Chung and Yau gave an example that showed their eigenvalue estimate $\lambda \geq \frac{1}{8dD^2}$ is a sharp one for Ricci-flat graph. If $m < \infty$, then we have a better estimate. For example, if G is a triangle graph with three vertices, then $m = 2$.

This is the key lemma to prove the functional inequalities in this section.

Lemma. Suppose G is a finite connected graph satisfying $CD(m, k)$, then for $x \in V$, we have

$$(\frac{4}{m} - 2)(\Delta f)^2 + (2 + 2k)|\nabla f|^2 \leq \frac{1}{d_x} \sum_{xy \in E} \frac{1}{d_y} \sum_{yz \in E} [f(x) - 2f(y) + f(z)]^2.$$

By using the above lemma, from the $CD(m, 0)$ condition, we can obtain

$$\Delta(|\nabla f(x)|^2) \geq 2\lambda|\nabla f(x)|^2 - \frac{4}{m}\lambda^2 f(x).$$

Meanwhile, we have

$$\Delta f^2(x) = 2\lambda f^2(x) - |\nabla f(x)|^2.$$

We can then use a maximum principle for function $|\nabla f|^2(x) + \alpha\lambda f^2(x)$ as Chung and Yau did.

For the proof of the corollary, we shall note that for the eigenfunction $f(x)$, we have $\sum_{x \in V} d_x f(x) = 0$.

3.2 Higher order eigenvalues

First we consider distance functions on vertices set V . We assume that we are given some distance function on V and denote it by $\rho_\xi(x)$. Denote by $B_\xi(x)$ the ball defined by ρ , that is

$$B_\xi(r) = \{x : \rho_\xi(x) < r\}$$

Let us assume that ρ_ξ has the following property:

$$|\nabla_{xy} \rho_\xi| \leq 1$$

for any edge $xy \in E$ and for any vertex $\xi \in V$, where

$$\nabla_{xy} f = f(y) - f(x)$$

is the value of the gradient of f assigns on each ordered pair $x, y \in V$.

Next, we will need the following constant characterizing a structure of edges at the boundary of the ball $B_\xi(r)$. Given points $\xi, x \in V$, consider the following sum of σ_{xy} over all points y adjacent to x and satisfying $\rho_\xi(y) < \rho_\xi(x)$:

$$\mu_x^{(\xi)} = \sum_{\{y: y \sim x \text{ and } \rho_\xi(y) < \rho_\xi(x)\}} \sigma_{xy}$$

Here σ_{xy} is the weight of edge xy . Clearly $\mu_x^{(\xi)} \leq \mu_x$, where μ_x is defined to be $\sum_{y: y \sim x} \sigma_{xy}$. We regard μ_x as a measure on vertices, namely for any subset Ω of vertices, $\mu(\Omega) = \sum_{x \in \Omega} \mu_x$.

We define the *spring ratio* ν_r , for any $r > 0$, as follows

$$\nu_r = \inf_{\xi \in M, x \in B_\xi(r)} \frac{\mu_x}{\mu_x^{(\xi)}}$$

Together with the function $\rho_\xi(x)$, we consider another function $q_\xi(x)$ - an analogue of the square distance. We postulate the following properties of q , for some positive constants δ, ι and R_0 :

1. $q_\xi(x) \geq 0$, and $q_\xi(x) = 0$ if and only if $x = \xi$.
2. For any vertex ξ and for arbitrary adjacent vertices $x, y \in B_\xi(R_0)$,

$$\nabla_{xy} q_\xi \leq \rho_\xi(x) + \iota;$$

clearly, we can always assume that

$$\iota \geq 1.$$

3. For any vertex ξ and all $x \in B_\xi(R_0)$,

$$\Delta q_\xi(x) \geq \delta.$$

Example. Let (Γ, σ) be the rectangular lattice graph defined on \mathbb{Z}^n . Let us consider

$$\rho_t(x) = \max_{1 \leq i \leq n} |x_i - \xi_i|$$

and

$$q_\xi(x) = \frac{1}{2} \sum_{i=1}^n (x_i - \xi_i)^2$$

In other word, $\rho_\xi(x)$ is the l^∞ -distance whereas $q_\xi(x)$ is determined by the l^2 -distance. It can be verified that condition 1 - condition 3 are true under these settings.

Definition. Given positive numbers δ, ι and R_0 , we say that a weighted graph (Γ, σ) has property $P(\delta, \iota, R_0)$ if there exist a distance function ρ satisfying $\nabla_{xy} \rho_\xi \leq 1$ and a function q satisfying the hypotheses 1 - 3 such that

$$n := \delta \nu_{R_0+1} \geq 1$$

where ν_{R_0+1} and δ which are the spring ratio and the parameter brought by the condition 3 above.

Theorem (Fan Chung, A. Grigoryan, S. T. Yau[7]). Assume that the weighted graph (Γ, σ) has property $P(n, \iota, R_0)$ and that

$$\omega' := \inf_{x \sim y} \sigma_{xy} > 0$$

and denote

$$\omega := \inf_{x \in V} \mu_x$$

Then for any finite set $\Omega \subset V$ the Dirichlet eigenvalue $\lambda_k(\Omega)$ satisfies

$$\lambda_k(\Omega) \geq a \left(\frac{k}{\mu(\Omega)} \right)^{\frac{2}{n}}$$

provided

$$|\Omega| \geq k \geq \kappa \frac{\mu(\Omega)}{\omega R_0^n}$$

where $\kappa = c_1(n) \iota^n > 0$ and $a = c_2(n) \iota^{-2} \nu_{R_0+1}^{-2} \omega'^2 \omega^{2/n-2} > 0$ and $|\Omega|$ is the number of vertices.

As a corollary we have

Theorem (Fan Chung, A. Grigoryan, S. T. Yau[7]). For the lattice graph on \mathbb{Z}^n , for any finite subset Ω of vertices, we have

$$\lambda_k(\Omega) \geq a \left(\frac{k}{\mu(\Omega)} \right)^{\frac{2}{n}}$$

where k is any integer between 1 and $|\Omega|$ and $a = a(n) > 0$.

4 Modified Ollivier's Ricci curvature on graphs and Ricci-flat graphs

4.1 Modified Ollivier's Ricci curvature

Recently, Lin-Lu-Yau[11] have modified the definition of Ollivier (for Ricci curvature of Markov chains on metric spaces[14]) to define Ricci curvature of a graph in the following way. A probability distribution is a function $m : V \rightarrow [0, 1]$ so that $\sum_{x \in V} m(x) = 1$. For any two probability distributions, m_1 and m_2 , the transportation distance is

$$W(m_1, m_2) = \sup_f \sum_{x \in V} f(x) [m_1(x) - m_2(x)]$$

where f is any Lipschitz function with constant one:

$$|f(x) - f(y)| \leq d(x, y)$$

Given $0 \leq \alpha \leq 1$, we define the probability distributions

$$m_x^\alpha(v) = \begin{cases} \alpha & \text{if } v = x ; \\ \frac{1-\alpha}{d_x} & \text{if } v \sim x ; \\ 0 & \text{otherwise .} \end{cases}$$

For any $x, y \in V$ define α -Ricci curvature

$$\kappa_\alpha(x, y) = 1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x, y)} .$$

Lemma 1. κ_α concaves upward for $0 \leq \alpha \leq 1$.

Lemma 2. $\kappa_\alpha(x, y) \leq (1 - \alpha) \frac{2}{d(x, y)}$.

When $\alpha = 0$, $\kappa_0(x, y)$ is Ricci curvature defined by Ollivier. We have the following theorem for $\kappa_0(x, y)$.

Theorem (Y. Lin, S. T. Yau[13]). The Ricci curvature of Ollivier $\kappa_0(x, y) \geq \frac{2}{d_x} + \frac{2}{d_y} - 2$ if $d_x > 1$ and $d_y > 1$; $\kappa_0(x, y) = 0$ if $d_x = 1$ or $d_y = 1$.

The lower bound can be achieved when the graph is a tree. This was showed by Jost and Liu[9] recently. They also find a relation between the Ricci curvature of Bakry-Emery and Ollivier on graphs.

We define

$$\lim_{\alpha \rightarrow 1} \frac{\kappa_\alpha(x, y)}{1 - \alpha} = \kappa(x, y)$$

to be the Ricci curvature for all pairs (x,y). From this definition, we know that

$$\kappa(x, y) \geq \kappa_0(x, y) .$$

Example. 1. The complete graph K_n has a constant Ricci curvature $\frac{n}{n-1}$ for every edge.

2. The cycle C_n for $n \geq 6$ has constant Ricci curvature 0 and

$$\kappa(C_3) = \frac{3}{2}, \quad \kappa(C_4) = 1, \quad \kappa(C_5) = \frac{1}{2}.$$

For graphs G and H , the Cartesian product $G \square H$ is a graph given by $V(G) \times V(H)$ and the two pairs (u_1, v_1) and (u_2, v_2) can be connected iff $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(G)$ and $v_1 = v_2$.

Theorem (Y. Lin, L.Y. Lu, S. T. Yau[11]). If G is d_G -regular and H is d_H -regular, then

$$\begin{aligned} \kappa^{G \square H}((u_1, v), (u_2, v)) &= \frac{d_G}{d_G + d_H} \kappa^G(u_1, u_2) \\ \kappa^{G \square H}((u, v_1), (u, v_2)) &= \frac{d_H}{d_G + d_H} \kappa^H(v_1, v_2). \end{aligned}$$

Remark. This theorem is not true if we replace $\kappa(x, y)$ by $\kappa_o(x, y)$. This is one of the advantage of our modified Ricci curvature.

Corollary. Suppose G is regular and has constant curvature κ . Then G^n , the n -th power of the Cartesian product of G , has constant curvature $\frac{\kappa}{n}$.

Bonnet-Myers type theorem for graphs.

Theorem (Y. Lin, L.Y. Lu, S. T. Yau[11]). If $\kappa(x, y) > 0$, then

$$d(x, y) \leq \left\lceil \frac{2}{\kappa(x, y)} \right\rceil.$$

If for any edge \bar{xy} , $\kappa(x, y) \geq c > 0$, then

$$\text{diam}(G) \leq \frac{2}{c},$$

Also, $\lambda_1 \geq c$, and

$$\begin{aligned} \text{Vol}(G) &= \text{number of vertices} \\ &\leq 1 + \sum_{k=1}^{\lfloor \frac{2}{c} \rfloor} \Delta^k \prod_{i=1}^{k-1} \left(1 - \frac{ic}{2}\right) \end{aligned}$$

where Δ is the maximum degree of G .

A random graph $G(n, p)$ is a graph on n vertices in which a pair of vertices appears as an edge with probability p .

For a random graph $G(n, p)$, we have

Theorem (Y. Lin, L.Y. Lu, S. T. Yau[11]). Suppose that xy is an edge of random graph $G(n, p)$. The following statements hold for the curvature $\kappa(x, y)$.

1. If $p \geq \sqrt[3]{(\ln n)/n}$, almost surely, we have

$$\kappa(x, y) = p + O\left(\sqrt{\frac{\ln n}{np}}\right).$$

In particular, if $p \gg \sqrt[3]{(\ln n)/n}$, almost surely, we have $\kappa(x, y) = (1 + o(1))p$.

2. If $\sqrt[3]{(\ln n)/n} > p \geq 2\sqrt{(\ln n)/n}$, almost surely, we have

$$\kappa(x, y) = O\left(\frac{\ln n}{np^2}\right).$$

3. If $1/\sqrt{n} \gg p \gg \sqrt[3]{(\ln n)/n^2}$, almost surely, we have

$$\kappa(x, y) = -1 + O(np^2) + O\left(\frac{\ln n}{n^2 p^3}\right).$$

4. If $\sqrt[3]{1/n^2} \gg p \gg \frac{\ln n}{n}$, almost surely, we have

$$\kappa(x, y) = -2 + O(n^2 p^3) + O\left(\sqrt{\frac{\ln n}{np}}\right).$$

Note that we say that a property P is almost surely satisfied if the limit of the probability that P holds goes to 1 as n goes to infinity.

4.2 Ricci-flat graphs

Ricci-flat manifolds are Riemannian manifolds with Ricci curvature vanishes. In Physics, they represent vacuum solutions to the analogues of Einstein's equations for Riemannian Manifolds of any dimension, with vanishing cosmological constance. The important class of Ricci-flat manifolds is Calabi-Yau manifolds. This follows from Yau's proof of the Calabi conjecture, which implies that a compact *Kähler* manifold with a vanishing first real Chern class has a *Kähler* metric in the same class with vanishing Ricci curvature. There are many works to find the Calabi-Yau manifolds. Yau conjectured that there are finitely many topological types of compact Calabi-Yau manifolds in each dimension. This conjecture is still open. In this paper, we will use our modified Ollivier's Ricci curvature study this question on graphs.

We recall the definition of Ricci curvature on graphs introduced by Fan Chung and Yau in 1996[6]. We say that a regular graph G has a local k -frame at a vertex x if there exist injective mappings η_1, \dots, η_k from a neighborhood of x into V so that

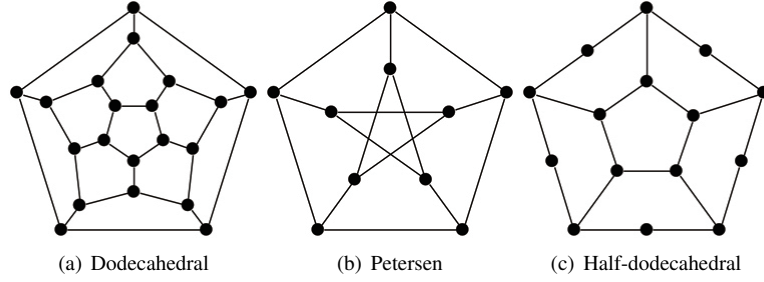
1. x is adjacent to $\eta_i x$ for $1 \leq i \leq k$;
2. $\eta_i x \neq \eta_j x$ if $i \neq j$

The graph G is said to be Ricci-flat at x if there is a local k -frame in a neighborhood of x so that for all i ,

$$\cup_j (\eta_i \eta_j) x = \cup_j (\eta_j \eta_i) x.$$

It is easy to show that the Ricci flat graphs defined by Chung and Yau are the Ricci-flat graphs in the sense of Ollivier's definition, and are the graphs with non-negative Ricci curvature of our modified definition.

The girth of a graph is the length of a shortest cycle contained in the graph.



The following theorem is our main result:

Theorem (Y. Lin, L.Y. Lu, S. T. Yau[12]). Suppose that G is a Ricci flat graph with girth $g(G) \geq 5$, then G is one of the following graphs:

1. the infinite line
2. cycle C_n with $n \geq 6$
3. the dodecahedral graph
4. the Petersen graph
5. the half-dodecahedral graph

Since the Cartesian product of two regular Ricci-flat graphs is still Ricci-flat, there are infinite number of Ricci-flat graphs with girth 4. Same happens for girth 3.

The homogeneous graph Γ associated with an abelian group \mathcal{H} is Ricci-flat in the sense of Chung and Yau. That means that \mathcal{H} is a subgroup of automorphism group of Γ and \mathcal{H} acts transitively on the vertex set V of Γ , i.e. for any two vertex u and v there is an $f \in \mathcal{H}$ such that $f(u) = v$. So there are tremendous number of Ricci-flat graphs in the sense of Chung and Yau and therefore Ricci-flat graphs in the sense of Ollivier. They are either girth 4 or 2. Our modified Ricci curvature are good definition to classify the Ricci-flat graphs.

5 Exterior forms on digraphs(A. Grigor'yan, Y. Lin, Y. Muranov, S. T. Yau[8])

5.1 homology and cohomology of digraphs

A differential calculus on an associative algebra A is an algebraic analogue of the calculus of differential forms on a smooth manifold. The discrete differential calculus is based on the universal differential calculus on an associative algebra of functions on discrete set. By a natural way, we can consider this calculus as a calculus on a universal digraph with the given discrete set of vertexes. This approach gives an opportunity to define differential calculus for every subgraph of the universal digraph.

Given a finite set V , we define a p -form ω on V as \mathbb{K} -valued function on V^{p+1} . The set of all p -forms is a linear space over \mathbb{K} that is denoted by Λ^p . It has a canonical basis $e^{i_0 \dots i_p}$. For any $\omega \in \Lambda^p$ we have

$$\omega = \sum_{i_0, \dots, i_p \in V} \omega_{i_0 \dots i_p} e^{i_0 \dots i_p}$$

where $\omega_{i_0 \dots i_p} = \omega(i_0, \dots, i_p)$. The exterior derivative $d: \Lambda^p \rightarrow \Lambda^{p+1}$ is defined by

$$(d\omega)_{i_0 \dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q \omega_{i_0 \dots \hat{i}_q \dots i_{p+1}}$$

and satisfies $d^2 = 0$, where the hat \hat{i}_q means omission of the index i_q .

We define a subspace $\mathcal{R}^p \subset \Lambda^p$ of regular forms that is spanned by $e_{i_0 \dots i_p}$ with regular paths $i_0 \dots i_p$ (when $i_k \neq i_{l+1}$), and observed that the spaces \mathcal{R}^p are invariant for d .

A p -path on V is a formal linear combination of the elementary p -paths $e_{i_0 \dots i_p} \equiv$

$i_0 \dots i_p$ and the linear space of all p -paths is denoted by Λ_p . For any $v \in \Lambda_p$ we have

$$v = \sum_{i_0, \dots, i_p \in V} v^{i_0 \dots i_p} e_{i_0 \dots i_p}$$

and a pairing with a p -path ω

$$(\omega, v) = \sum_{i_0, \dots, i_p} \omega_{i_0 \dots i_p} v^{i_0 \dots i_p}$$

The dual operator $\partial : \Lambda_{p+1} \rightarrow \Lambda_p$ is given by

$$\partial e_{i_0 \dots i_{p+1}} = \sum_{q=0}^{p+1} (-1)^q e_{i_0 \dots \hat{i}_q \dots i_{p+1}}$$

Let I_p be the subspace of Λ_p that is spanned by $e_{i_0 \dots i_p}$ with irregular paths $i_0 \dots i_p$. Then the spaces I_p are invariant for ∂ , which allows to define ∂ on the quotient spaces $\mathcal{R} = \Lambda_p / I_p$. For simplicity of notation we identify the elements of \mathcal{R}_p with their representatives that are regular p -paths. Then $e_{i_0 \dots i_p}$ with irregular $i_0 \dots i_p$ are treated as zeros.

A digraph is a pair of (V, E) where V is an arbitrary set and E is a subset of $V \times V \setminus \text{diag}$. The elements of V are called *vertices* and the elements of E are called (*directed*) *edges*. The set V will be always assumed non-empty and finite.

Let $i_0 \dots i_p$ be an elementary regular p -path on V . It is called *allowed* if $i_k i_{k+1} \in E$ for any $k = 0, \dots, p-1$, and *non-allowed* otherwise. The set of all allowed elementary p -paths will be denoted by E_p , and non-allowed by N_p . For example $E_0 = V$ and $E_1 = E$.

Denote by $\mathcal{A}_p = \mathcal{A}_p(V, E)$ the subspace of \mathcal{R}_p spanned by the allowed elementary p -paths, that is,

$$\begin{aligned} \mathcal{A}_p &= \text{span}\{e_{i_0 \dots i_p} : i_0 \dots i_p \in E_p\} \\ &= \{v \in \mathcal{R}_p : v^{i_0 \dots i_p} = 0 \ \forall i_0 \dots i_p \in N_p\} \end{aligned}$$

The elements of \mathcal{A}_p are called allowed p -paths.

Similarly, denote by \mathcal{N}^p the subspace of \mathcal{R}^p , spanned by the non-allowed elementary p -forms, that is,

$$\begin{aligned}\mathcal{N}^p &= \text{span}\{e^{i_0 \dots i_p} : i_0 \dots i_p \in N_p\} \\ &= \{\omega \in \mathcal{R}^p : \omega_{i_0 \dots i_p} = 0 \ \forall i_0 \dots i_p \in E_p\}\end{aligned}$$

Clearly, we have $\mathcal{A}_p = (\mathcal{N}^p)^\perp$ where \perp refers to the annihilator subspace with respect to the couple $(\mathcal{R}^p, \mathcal{R}_p)$ of dual spaces.

We would like to reduce the space \mathcal{R}^p of regular p -forms so that the non-allowed forms can be treated as zeros. Consider the following subspaces of spaces \mathcal{R}^p

$$\boxed{\mathcal{J}^p \equiv \mathcal{J}^p(V, E) := \mathcal{N}^p + d\mathcal{N}^{p-1}}$$

that are d -invariant, and define the space Ω^p of p -forms on the digraph (V, E) by

$$\boxed{\Omega^p \equiv \Omega^p(V, E) := \mathcal{R}^p / \mathcal{J}^p}$$

Then d is well-defined on Ω^p and we obtain a cochain complex

$$0 \rightarrow \Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1} \xrightarrow{d} \dots$$

Shortly we write $\Omega = \mathcal{R}/\mathcal{J}$ where Ω is the complex and \mathcal{R} and \mathcal{J} refer to the corresponding cochain complexes.

If the digraph (V, E) is complete, that is, $E = V \times V \setminus \text{diag}$ then the spaces \mathcal{N}^p and \mathcal{J}^p are trivial, and $\Omega^p = \mathcal{R}^p$.

Consider the following subspaces of \mathcal{A}_p

$$\boxed{\Omega_p \equiv \Omega_p(V, E) = \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}}$$

that are ∂ -invariant. Indeed, $v \in \Omega_p \Rightarrow \partial v \in \mathcal{A}_{p-1} \subset \Omega_{p-1}$. The elements of Ω_p are called ∂ -invariant p -paths.

We obtain a chain complex Ω

$$0 \longleftarrow \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \dots$$

that, in fact, is dual to Ω .

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$ so that

$$\dim \Omega_0 = |V| \quad \text{and} \quad \dim \Omega_1 = |E|,$$

while in general $\Omega_p \subset \mathcal{A}_p$

Let us define the (co)homologies of the digraph (V, E) by

$$H^p(V, E) := H^p(\Omega) \quad \text{and} \quad H_p(V, E) := H_p(\Omega)$$

Recall that $H^p(V, E)$ and $H_p(V, E)$ are dual and hence their dimensions are the same. The values of $\dim H_p(V, E)$ can be regarded as invariants of the digraph (V, E) .

Note that for any $p \geq 0$

$$\dim H_p(\Omega) = \dim \Omega_p - \dim \partial \Omega_p - \dim \partial \Omega_{p+1}$$

Let us define the Euler characteristic of the digraph (V, E) by

$$\chi(V, E) = \sum_{p=0}^n (-1)^p \dim H_p(\Omega)$$

provided n is so big that

$$\dim H_p(\Omega) = 0 \quad \forall p > n$$

Example. Consider the graph of 6 vertices $V = \{0, 1, 2, 3, 5\}$ with 8 edges $E = \{01, 02, 13, 14, 23, 24, 53, 54\}$.

The image $\partial\Omega_2$ is spanned by two 1-paths

$$\begin{aligned}\partial(e_{013} - e_{023}) &= e_{13} - e_{03} + e_{01} - (e_{23} - e_{03} + e_{02}) \\ &= e_{13} + e_{01} - e_{23} - e_{02} \\ \partial(e_{014} - e_{024}) &= e_{14} - e_{04} + e_{01} - (e_{24} - e_{04} + e_{02}) \\ &= e_{14} + e_{01} - e_{24} - e_{02}\end{aligned}$$

that are clearly linearly independent. Hence, $\dim\partial\Omega_2 = 2$ whence $\dim H_2 = 0$.

The dimension of H_1 can be computed similarly, but we can do easier using the Euler characteristic:

$$\dim H_0 - \dim H_1 + \dim H_2 = \chi = 0$$

whence $\dim H_1 = 1$.

In fact, a non-trivial element of H_1 is determined by 1-path

$$v = e_{13} = e_{14} - e_{53} + e_{54}$$

Indeed, by a direct computation $\partial v = 0$, so that $v \in \ker \partial|_{\Omega_1}$ while for v to be in $\text{Im} \partial|_{\Omega_2}$ it should be a linear combination of $\partial(e_{013} - e_{023})$ and $\partial(e_{014} - e_{024})$, which is not possible since they do not have the term e_{54} .

We can do some transformations of digraphs to get the homology and cohomology group for new graphs.

We can define the Hodge Laplacian on p -forms, and study the spectrum of such Laplacian. There are interesting properties of these spectrum, such as the torsion defined by them, that can be developed parallel to Riemannian geometry.

Naturally these spectrum are invariants of the graph that can give a great deal of information about the graph.

5.2 Minimal paths and hole detection

The elements of $H_p(V, E)$ can be regarded as p -dimensional holes in the digraph (V, E) . To make this notion more geometric, we can work with representatives of the homologies classes, which are closed p -paths. We say that two closed p -paths u and v are homological and write $u \sim v$ if u and v represent the same homology class, that is, if $u \sim v$ is exact.

For any p -forms v define its length by

$$l(v) = \sum_{i_0, \dots, i_p \in V} |v^{i_0 \dots i_p}|$$

Given a closed p -path v_0 , consider the minimization problem

$$l(v) \mapsto \min \quad \text{for } v \sim v_0$$

This problem always has a solution, although not necessarily unique. Any solution is called a *minimal p -path*. It is hoped that minimal p -paths(in a given homology class) match our geometric intuition of what holes in a graph should be. We can give some examples of minimal paths to support this claim. The following is one of the example in dimension 2.

Example. Consider a digraph on Figure 2.

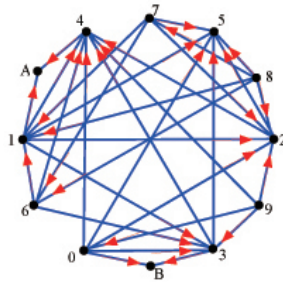


Figure : 2

Removing successively the vertices $A, B, 8, 9, 6, 7$ we obtain a digraph (V', E') with $V' = \{0, 1, 2, 3, 4, 5\}$ and $E' = \{02, 03, 04, 05, 12, 13, 14, 15, 24, 25, 34, 35\}$ that has the same homologies as (V, E) . The digraph (V', E') is shown in two ways on the following figure. Clearly the second representation of this graph is reminiscent of an octahedron.

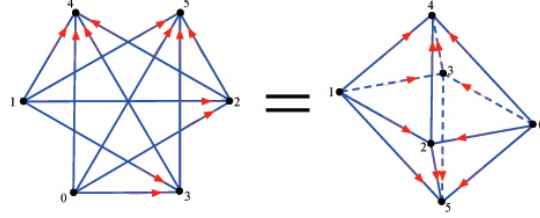


Figure : 3

The digraph (V', E') is the same as the 2-dimensional sphere-graph. Hence we obtain that $\dim H_2(V, E) = 1$ while $H_p(V, E) = 0$ for $p = 1$ and $p > 2$.

Consider a closed 2-path on (V, E)

$$\begin{aligned} v_0 = & e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} \\ & + e_{134} - e_{135} - e_{634} + e_{614} - e_{613} \end{aligned}$$

Then a solution to the minimization problem is given by

$$v = e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135}$$

that is a 2-path that determines a 2-dimensional hole in (V, E) given by the octahedron. Note that on Figure 2 this octahedron is hardly visible, but it can be computed purely algebraically as shown above.

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